Generalized Biform Games

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Abstract

How to extend the use of value-based strategy models to situations with large quasi-rents shared among multiple actors, such as ecosystems? How to consider how players understand competition in value-based models? How to overcome some limitations of these models such as lack of uniqueness of solutions? In this paper, we extend the reach of value-based strategy by revisiting the celebrated biform games model to answer these questions. Operationally, we make players evaluate their payoff from the cooperative stage of the game according to a generalized expectation over their value capture. Our solution has several advantages: (i) It subsumes the original biform framework and seamlessly integrates recent works providing bounds to value capture (ii) It allows solving issues such as the possible non-uniqueness of solutions and invariance to the competitive environment structure while maintaining the role of competition in determining value capture (iii) It remains axiomatically justified on behavioral grounds (iv) It permits richer preferences representations that, for example, can include subjective distortions of objective chances of value capture (v) It further leads the way to the use of generalized preference representations in the value-based framework.

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1 Introduction

The study of the relations between value creation and value capture and their implications on firms' behavior is central to the modern analysis of competitive strategy. These treatments have their theoretical bases in the seminal work of Brandenburger and Stuart (1996; 2007) which structure the problem of value creation and appropriation in an hybrid game theory setup. In particular, Brandenburger and Stuart (2007) propose a procedure for evaluating multistage games which have a non-cooperative stage followed by a cooperative stage (*biform games*). Their procedure prescribes that every player backward inducts his payoff from the cooperative stage and *evaluates it as a convex combination of the extremes of that player's coordinate projection on the core* (Gillies, 1953), the intervals of value capture considered in their work. In other words, they posit that players equate the value of each cooperative stage sub-game to the weighted combination of the maximum and minimum outcome that a player can obtain in the core of each sub-game.

This work takes the hybrid noncooperative-cooperative biform game setup of Brandenburger and Stuart (2007) and enriches its structure. Our technical generalization allows not only to access the use of different payoff evaluation criteria that are able to represent a variety of behavioral attitudes, but also for the embedding of uncertainty. In particular, we explore the implications of using an evaluation criterion that features an additional element with respect to that in Brandenburger and Stuart (2007): the expected value that a given player can capture in the cooperative game. This expectation is assessed by the player by considering the relative frequency with which a given allocation is assigned to him, among those possible, conditional on the full competitive structure of the cooperative game that is played. From a strategic management viewpoint, this formulation, that embeds as a special case the original biform games' structure, allows to incorporate the effects of business strategies on the shape of the competitive environment while leaving unconstrained the free-form competition that characterizes the cooperative stage of the game.

Finally, in line with the stream of research that suggests the use of alternatives to the core, the setup we propose is a framework that allows for the use of different solution concepts. In fact, the issue of how to evaluate of the cooperative-stage payoffs is conceptually separated from the one of which cooperative-stage payoffs to evaluate. Most of the analysis that we will carry out in the present paper is developed with a focus on the use of the core. However, we do not take a normative stance as to which solution concept to prescribe since its adequacy depends upon the assumptions that players are required to satisfy in a given strategic situation, as we will further discuss.

2 Background and Positioning

Recent work in strategic management considers the use of alternative solution concepts for the cooperative stage of biform games (Gans and Ryall, 2017; Ross, 2018) and their relative advantages and disadvantages with respect to the one proposed by Brandenburger and Stuart (2007), who focus on the core.

One of the main critiques to the use of the core in value-based business strategy relates to its possible non-existence. Dealing with this issue has been the focus of analysis of a series of papers that studied either different intervals of value appropriability (MacDonald and Ryall, 2004; 2018; Montez et al, 2017) or general conditions for the existence of the core in classes of games of particular interest to strategy scholars (Chatain and Zemsky, 2007; Stuart, 1997; 2004). Relative to the issue of non-uniqueness of the equilibrium, which is salient for the core, Ross (2018) proposes, for multi-player settings, the use of alternative point solutions, most notably the Shapley value (Shapley, 1962; Gale and Shapley; 1953) and the nucleolus (Schmeidler, 1969). Yet, putting aside the computational burden that they entail, these point solutions have features that may be undesirable in a strategic management setting. For instance, the Shapley value may not abide to minimal requirements of rationality in strategic thinking for large classes of games as it may allocate value to some players while others would have the ability to block them from capturing it, whereas the nucleolus can be interpreted as a "pessimistic" point solution.[1]

The use of these point solutions has also often been the answer to the problem of non-existence of the core, where a single-valued solution is proposed as an alternative to the

[1]The structure of the nucleolus solution is indeed reminiscent of the max-min criterion axiomatized in Gilboa and Schmeidler (1989).

use of the core when the latter is empty. As we argue more at length in Section 5.1, we remark that emptiness of the core is equivalent to the impossibility of allocations to satisfy a number of strategic reasoning requirements. Taking stock of this observation, we illustrate other solution concepts that are closer in spirit to the core and their relationship with it.

Another set of issues comes from the approach taken by Brandenburger and Stuart (2007), who provide a point solution to the problem of evaluating cooperative-stage payoffs when the core does not provide a unique allocation. This is one of the main virtues of their apparatus but it comes at a cost. Indeed, the use of a convex combination of the extremes of core projections, as already noted by the authors themselves (Brandenburger and Stuart, 2007: Appendix B.ii), implies that *a player doesn't distinguish between two cores that yield the same projection for that player*. This feature of the original biform games setup can be problematic in some applied contexts where very different strategic situations can give rise to similar core projections, although intuition suggests that a reasonable decision maker should not be indifferent between these situations.

This is, for example, the case in business ecosystems where value can be created as the result of complementarities between several types of players, all necessary to value creation. To fix ideas, let us consider a situation in which three types of players are necessary for value creation (e.g. hardware manufacturers, operating systems, and application software) and that there is only one player of each kind. Each player's added value is equal to the total value created and the projection of the core for each player is an interval between zero and the total value of the game. In the classic Brandenburger and Stuart (2007) framework, each player evaluates this as a combination of the best and worst points in that interval, not necessarily egalitarian. Now, assume that a new hardware manufacturer enters the game, and is a perfect substitute (clone) to the one already in the game. Obviously, value captured by the hardware manufacturers drops to zero due to competition. However, in this new game, the upper and lower bounds for value appropriation of the operating system and the application software remain similar to those in the original game and, according to the classic framework, their envisaged value capture should remain the same. Yet, before entry, the full value had to be split three ways, while after entry, it only has to be split between two players who, intuitively, would probably prefer the second game to the first. This is because while the projection of the core does not change for each of these two players, the geometry of the set of core allocations is different in the second game, and arguably more favorable to them, as it now assigns zero value capture to the manufacturers while initially the unique manufacturer could appropriate a wide range of value. This highlights a substantial problem for the applicability of the classic biform games framework in modeling such situation: it seems impossible to recognize the effect of the time-honored strategy of commoditizing complementors in ecosystems in order to capture more value. Conceptually, this example suggests that it may be useful to base the evaluation of value capture on the full set of value capture possibilities across all players, rather than just on the projection of these along one single axis.

In this paper we aim at addressing the issues raised above. In particular, we propose a generalization of the biform games of Brandenburger and Stuart (2007) which embeds richer implications of the structure of the competition on the chances of appropriation of value for each player. Operationally, this translates into players evaluating their payoff from the cooperative stage of the game by an expected value of their value capture that takes into account the full structure of the constraints on value capture, in addition to the upper and lower bounds on value capture.

For set-valued solution concepts, this evaluation can be decomposed into two fundamental constituents. The first one is objective and is given by the expected value of the interval of value capture, where the probability of each possible value captured by a player is given by the objective chances that such value is allocated to the player, given the structure of the game. In other words, each player, when considering the range of possible values that he can capture when joining a coalition, weighs these values by the relative share of allocations that assign such value to him. The other component instead allows to incorporate subjective distortions of such objective expected value by incorporating subjective weights over the lower and the upper bounds of value appropriation.

To this end, we provide an extended definition of biform games, in which we replace the original confidence index based evaluation criterion of Brandenburger and Stuart (2007) with a more general preference relation over payoffs,[2] that is also backward compatible with

[2] We observe that our approach is different from that of Agastya (1996), where the problem studied is

the latter seminal biform games criterion. Relatedly, our embedding of a generic structure of preferences in the biform game definition leads the way to the use of richer payoff evaluation criteria, as advocated by Gans and Ryall (2017).

Our solution has several advantages. First, it embeds the biform games' structure of Brandenburger and Stuart (2007) as a special case, and thus preserves all of their results when we restrict to their framework. In addition, our formal structure allows to consider more general sets of outcomes, that do not necessarily coincide with the core. In particular, we can apply our results to the recent developments in the value-based strategy literature that focused on the implications of different strategic requirements on intervals of value capture (MacDonald and Ryall, 2004; Montez, Ruiz-Aliseda, and Ryall, 2017). Finally, considering more general outcome sets makes it possible to reconcile the use of a point-selection from a set of constraints on value capture with the use of the Shapley value, under an overarching setup which frames both choices as special cases of a unified evaluation criterion. Finally, different strategic constraints on the solution set can further be interpreted as different limits to strategic cognition — more elaborate cognitive capacities correspond to more requirements and constraints. Specifically, this helps translating explicitly the impact of bounded cognition in strategic reasoning over value creation into different perceived intervals of value capture.

The setup presented in this paper is therefore very flexible and can be tuned to study a variety of meaningful situations. In what follows, we will focus on the case that corresponds to assigning equal chances to allocations that are solutions of the cooperative stage game, as this assumption naturally descends from the use of set valued solution concepts like the core. However, as we will discuss later, our setup allows for the use of more general assumptions on the probabilistic structure imposed on the set of allocations. This work thus represents only a first step in the direction of analyzing the implications of using a richer, yet still parsimonious, structure in biform games.

that of choice under uncertainty between bargaining situations.

3 Examples

The following examples illustrate how the use of a refined evaluation criterion for the cooperative stage leads the way to novel behavioral insights together with an embedding of the effects of competition on strategic choices.

Example 1. *"Invariance to the competitive structure: Wide cores are not equivalent"*

We reformulate and elaborate on the example developed in the introduction. Let's consider a status-quo cooperative game in which there are three players: a supplier of operating system A, an hardware manufacturer B, and a developer of application software C. All together they can create a total value of \$3, but no value can be created by any other subset of players. This status-quo game has non-empty core and each player has the same added value, which is equal to \$3. Hence, the lower bound on value creation for each player is \$0 while the upper bound is \$3.

Let us now assume that the operating system provider A can help another hardware manufacturer D entering the game at no extra cost. This case corresponds to a different coalitional game. Let us assume that this game is such that the total value created does not vary as the hardware clone joins the coalition of all players. Now all coalitions that are composed by the operating system, one or two hardware manufacturer, and the application software developer produce a total value of \$3 while all other coalitions create no value. In the core of this game, the operating system can capture value between \$0 and \$3, the application software developer can capture between \$0 and \$3, and the manufacturers (original and clone) capture \$0, as they are identical and have no added value individually.

From the view point of the operating system provider, the upper and lower bounds on value capture in the new game are identical to those in the status-quo game. Following Brandenburger and Stuart (2007), this implies that, for fixed confidence indices, the manufacturer should be indifferent between helping or not helping the clones enter the market. Yet, in the standard biform games' setup no direction is given regarding the determinants of the confidence index. In particular, it does not need to vary with the shape of the core.

The simple but crucial observation in this case is that, although the range of possible values that the manufacturer can capture in this game did not change, there is a change in what can be seen as the manufacturer's objective chances of value appropriation. This consideration can be translated here by computing the expected allocation vector for all players (that, as we will remark, corresponds to the core-center of the status-quo game), which prescribes that on average a value of \$1 is allocated to each player. Therefore, according to expected value reasoning, the operating system provider foresees to appropriate, on average, a value of \$1 in the status quo game.

In the new coalitional game, only two players, namely the manufacturer and the buyer, can appropriate positive value since they are the only two players whose added value is non-zero. This implies that, after the clone enters the market, value will be shared among fewer players. In this case, the operating system's expected value capture increases from \$1 to $\$\frac{3}{2}$, making the action of helping the entrance of additional suppliers in the market preferred to the status-quo, from the manufacturer's point of view, in accordance with intuitive reasoning. □

Bringing in substitutes to complementors (or *commoditizing*) is a widespread strategy in ecosystems. Yet, the classic framework cannot always capture the rationale for this strategy. Our simple example shows how expected allocation considerations help embedding considerations about the competitive structure of the game also in those cases in which they do not imply differences in the intervals of value capture for some players in different coalitional games. We finally remark that, in this case, the Shapley value of the 4-players game would give some strictly positive value is allocated to the hardware manufacturers notwithstanding the fact that their added value is null.

The next example is given to illustrate the possibility of disentangling an objective component of the evaluation of the cooperative stage payoffs from a subjective one, that is determined by confidence considerations. Abstracting from behavioral arguments, the following example shows that our solution also addresses critiques related to the set-valued nature of the core.

Example 2. *"A behavioral twist on Brandenburger and Stuart (2007)"*

Let us revisit the biform analysis of the "branded ingredient" strategy in Brandenburger and Stuart (2007). In this game, there are two firms such that each can produce a single unit of a given product. In the economy, there is also only one supplier that can supply the input to at most one firm, and the cost of this necessary input is $1. Finally, there are numerous buyers, each demanding at most one unit of product, from either of the two firms. Buyers all have the same tastes but the value that they attach to the product is firm-specific. This situation describes, for example, the case in which two firms produce the same product but in different qualities. It is assumed that buyers are willing to pay up to $9 for the product sold by Firm 1 and up to $3 for the product sold by Firm 2. It is further assumed that the supplier has the option of incurring an upfront cost of $1 to increase the buyers' willingness to pay for Firm 2's product up to $7, by branding the product of Firm 2. This situation can be modeled as a biform game in which the supplier is the only player who can move in the (non-cooperative) first-stage of the game and can choose whether or not to incur the upfront cost for branding the product of Firm 2.

It is easy to observe that, in both second-stage (cooperative) games, the buyers and Firm 2 have zero added value. If we focus on the use of the core as a solution concept for our cooperative sub-games, then this prescribes that neither the buyers nor Firm 2 is ever able to appropriate positive value. The situation is different for Firm 1 and the supplier: in the status-quo sub-game, Firm 1 can appropriate value in a range that goes from a minimum of $0 up to a maximum of $6 while the supplier can appropriate value in a range that goes from a minimum of $2 to a maximum of $8. Whereas, in the branded-ingredient sub-game, Firm 1 can appropriate value in a range that goes from a minimum of $0 up to a maximum of $2 and the supplier can appropriate value in a range that goes from a minimum of $5 to a maximum of $7.

Although the supplier can secure at least $2 in the status-quo game and $5 in the branded ingredient strategy, the relative preference for one strategy over the other is completely determined, in Brandenburger and Stuart (2007), by the players' confidence indices and nothing can be said ex-ante. On the sole basis of payoffs intervals, similar considerations apply to Firm 1, if one is not to invoke exogenous bargaining power arguments which may not, however, always apply to multilateral bargaining settings.[3]

[3]Previous work using biform games, starting with Chatain and Zemsky (2008) have interpreted the α_i coefficient as an indicator of bargaining power, notably between buyers (e.g. endowed with $\alpha_i = \alpha$ bargaining strength) and suppliers (with $\alpha_i = 1-\alpha$ bargaining strength). This interpretation has the advantage of fitting

Reasoning in terms of expected appropriation within the core, instead, offers us a way out of this indeterminacy, given that in the status-quo sub-game, the average allocation in the core is \$5 for the supplier and \$3 for Firm 1, whereas, in the branded strategy sub-game, the supplier obtains on average \$6, whereas Firm 1 is allocated \$1, on average. □

4 Formal definitions and preliminaries

In this section we lay out the mathematical structures and definitions that we will use throughout the paper. Let us start by considering a multi-stage game where the first-stage is a simultaneous-moves game while the second-stage is a cooperative transferable-utility game. All the definitions below can be extended to the case where there is a multiplicity of non-cooperative stages before the last, cooperative, stage.

4.1 Formal definitions of games

A *simultaneous-moves game* is represented by the tuple

$$\mathbb{G} = \langle I, Y, (A_i)_{i \in I}, g, (v_i)_{i \in I} \rangle$$

where

- $I = \{i, i = 1, ...n\}$ is the set of *players*. For every player $i \in I$, we denote by $-i$ all the other players except i;
- A_i is the nonempty set of *actions* for player $i \in I$. We denote by $a_i \in A_i$ the generic action of player i, hence $A = A_i \times A_{-i}$ denotes the set of available actions and elements of A are arrays $a = (a_i, a_{-i})$;

with notions of bargaining power used in classic strategy frameworks of industry analysis, and especially Porter's five forces (Porter, 1980). However, this interpretation, while adequate in bilateral negotiations environments, loses its bite in situations when bargaining over value involves more than two sides at the same time.

- $g : \times_{i \in I} A_i \to Y$ is the *consequence function*, which maps action profiles into consequences $g(a) \in Y$ and captures the essence of the rules of the game, beyond the assumption of simultaneous moves;

- $v_i : Y \to \mathbb{R}$ is the *von Neumann-Morgenstern utility function* of player i.

From the consequence function g and the utility function v_i of player i, we obtain a function that assigns to each $a = (a_j)_{j \in I}$ the utility $v_i(g(a))$ for player i of consequence $g(a)$. This function

$$u_i = v_i \circ g : \times_{i \in I} A_i \to \mathbb{R}$$

is called the *payoff function* of player i.

In order to make explicit the dependence of the second stage game upon the action profile chosen at the first stage we introduce a notion of conditional cooperative game.

A *conditional*[4] *transferable utility (TU) cooperative game evaluated in the action profile* a is a pair

$$G|a = \langle I, W_a \rangle$$

where $W_a : 2^I \to \mathbb{R}$ is the characteristic function of the conditional TU game. It is defined as the section at a of the function $W : 2^I \times A \to \mathbb{R}$ that assigns to every coalition $C \subseteq I$ and action profile $a \in A$ its worth $W(C, a)$. By convention, we set $W(\emptyset, a) = 0$ for all $a \in A$ and we assume that $W(\{i\}, a) = W(\{i\}, a')$ for all $a, a' \in A$. Finally, we denote by $\mathcal{G}^n$ the set of n-player (conditional) TU games.

We say that a conditional TU game $G|a$ is *zero-normalized* if and only if $W_a(\{i\}) = 0$ for each $i \in I$. Whereas, a conditional TU game $G|a$ is *superadditive* if and only if, for all $S, T \subseteq I$ it holds that $W_a(S) + W_a(T) \leq W_a(S \cup T)$ and it is *convex* if and only if, for all $S, T \subseteq I$ it holds that $W_a(S) + W_a(T) \leq W_a(S \cup T) + W_a(S \cap T)$.

We are now ready to give the definition of our class of biform games.

[4] Although this terminology is not used in Brandenburger and Stuart (2007), there is no conceptual difference between the second stage cooperative games considered in their work and those studied here.

Definition 1. A *generalized biform game* is a tuple

$$\mathbf{\Gamma} = \left\langle I, (A_i)_{i \in I}, (\succsim_i^*)_{i \in I}, V \right\rangle$$

where

- $I = \{i : i = 1, ..., n\}$ is a set of players;
- A_i is the nonempty set of possible *actions* for player $i \in I$;
- $\succsim_i^*$ denotes player i's preference relation over (cooperative-stage) outcomes;
- $V : A \to \mathbb{R}^{2^I}$ is the value function of the biform game and assigns, to every action profile $a \in A$ the *value* of every coalition $C \subseteq I$ in the cooperative stage of the game.

The value function of the biform game allows to link the non-cooperative stage with the cooperative stage of the game as follows: for every $C \subseteq I$, we set $V(a)(C) = W(C, a)$, and therefore $V(a)$ corresponds to W_a, that is the section at a of the characteristic function of the conditional TU game $G|a$.

4.2 Value Creation and Allocations of Value

The value created in the conditional cooperative stage game is distributed among the players. A distribution of value (*allocation*) for the conditional game $G|a \in \mathcal{G}^n$ is a profile $\pi^a = \left(\pi_j^a\right)_{j \in C} \in \mathbb{R}_+^n$ where π_j^a is a real number indicating the amount of *value captured* by agent j in return for his participation in the value-creating activities that contribute to the production of $W_a(C)$, the *value created* by coalition C in the conditional game $G|a$. In particular, $W_a(\{i\})$ denotes the value that player i can create on his own.

The *added value of player* i to a coalition C is defined as

$$av_i(C; a) := W_a(C) - W_a(C \setminus \{i\})$$

that is the difference between the value that the coalition C can create when i belongs to the coalition and the value that the coalition C can create when i does not belong to the coalition, when action profile a is chosen (in the non-cooperative stage).

Allocation can be characterized by properties with respect to how the value created is distributed among the players. In particular, we say that an allocation π^a is *efficient* (or *feasible* in Gans and Ryall, 2017) if $\sum_{i\in I}\pi_i^a = W_a(I)$, i.e. if all the value created by the full set of players is distributed among the players of the game; π^a is *individually rational* if, for each $i \in I$, $\pi_i^a \geq W_a(\{i\})$, i.e. if each player is allocated at least the value that he could create on his own; whereas π^a is *coalitionally rational* if, for every non-empty coalition $C \subseteq I$, $\sum_{i\in C}\pi_i \geq W_a(\{i\})$.[5] Relatedly, the *imputation set* of $G|a \in \mathcal{G}^n$ is the set of all efficient and individually rational allocations for the conditional game $G|a \in \mathcal{G}^n$, i.e.

$$\mathcal{I}(a) := \left\{\pi^a \in \mathbb{R}^n : \sum_{i\in I}\pi_i^a = W_a(I)\,, \text{and for each } i \in I, \pi_i^a \geq W_a(\{i\})\right\}$$

and the elements $\pi^a \in \mathcal{I}(a)$ are called *imputations*. Instead, we will refer to the set of efficient allocations for a conditional game $G|a \in \mathcal{G}^n$ as the *pre-imputation set* of $G|a$. Geometrically, the imputation set of a zero-normalized coalitional n-player game can be represented, w.l.o.g., by an $(n-1)$-dimensional standard simplex.[6] Since the techniques we develop here rely on properties of simplices, in what follows we will focus mainly on allocations that are non-negative and efficient. However, as we discuss later on, this fact does not constitutes a limitation to our framework due to the possibility of re-normalizing payoffs in conditional games, as already noted in Brandenburger and Stuart (2007) and Chatain and Zemsky (2011), respectively.

Finally, we consider basic monotonicity properties of allocations that will be useful in order to characterize outcome sets of the cooperative-stage game. Let π^a, ρ^a be allocations and let $S \subsetneqq I$ be a nonempty coalition. We say that π^a dominates ρ^a via S if $\pi_i^a > \rho_i^a$ for all $i \in S$ and $\sum_{i\in S}\pi_i^a < W_a(S)$ and we say that π^a dominates ρ^a if there is an $S \subset I$ such that π^a dominates ρ^a via S. For an imputation π^a, let $Dom(\pi^a)$ be the set of imputations

[5]Note that Gans and Ryall (2017) refer to coalitional rationality as *competitive consistency* (or *stability*) condition. Competitive consistency trivially implies individual rationality.

[6]The m-dimensional standard simplex Δ^m is the convex envelope of the canonical base $e_1, ..., e_{m+1}$ of $\mathbb{R}^{n+1}$. That is, $\Delta^m = \left\{(x_1, ..., x_{m+1}) \in \mathbb{R}^{m+1} : \sum_{i=1}^{m+1} x_i = 1, x_i \geq 0\ \forall i\right\}$. Note that Δ^m is a regular simplex, i.e. a simplex such that the distance between any two of its vertices is constant.

dominated by π^a, and for a set of imputations $K \subsetneqq \mathcal{I}(a)$, let $Dom(K) = \bigcup_{\pi^a \in K} Dom(\pi^a)$ be the set of imputations which are dominated by at least one imputation in K.

4.3 Value Capture and Probabilistic Solutions

Let us denote by $\chi_i\,(C;a)$ the *interval of value capture* (or *appropriability interval*) for player $i \in I$, that is, the set of values π_i^a that player i can capture when coalition C is formed and the action profile a has been played in the non-cooperative stage. Further denote by I^+ the set of players such that $\chi_i\,(C;a) \neq \{0\}$ for all $i \in I^+ \subseteq I$. Different strategic requirements have different implications on the definition of $\chi_i\,(C;a)$.

For example, it is well known that, under regularity conditions,[7] individual rationality and efficiency imply $W_a\,(\{i\}) \leq \pi_i^a \leq av_i(C;a)$. Hence, minimal strategic requirements suggest the definition of the following candidate interval of value capture:

$$\chi_i^0(I;a) = [W_a\,(\{i\})\,, av_i(I;a)]$$

and we refer to it as the *added value capture interval* of player $i \in I$.[8]

Arguably, the most commonly used type of interval of value capture in value based business strategy is given by the projections onto the i-th coordinate axis of the core of the conditional cooperative games. We recall that the core of a conditional TU game is the set of imputations that are coalitionally rational, that is

$$\mathcal{C}\,(G|a) = \left\{ \pi^a \in \mathbb{R}^n : \sum_{i\in I} \pi_i^a = W_a\,(I)\,, \text{and for every } \emptyset \neq C \subseteq I, \sum_{i\in C} \pi_i^a \geq W_a\,(C) \right\}.$$

Then we can denote the resulting *interval of core value capture* by

$$\chi_i^\infty(I;a) = \mathrm{proj}_i\,\{\mathcal{C}\,(G|a)\}\,.$$

[7]In particular, let $G|a \in \mathcal{G}^n$ be a zero-normalized conditional TU game such that $av_i(I;a) \geq 0$ for all $i \in I$.

[8]Building on the logic underlying the added value principle, other relevant examples of bounds can be found, among others, in Tijs (1981), MacDonald and Ryall (2004) and Montez et al. (2017). As anticipated in the introduction, we can extend our analysis to the use of these bounds.

Yet, the core is only one instance of solution concept for cooperative games. A *probabilistic solution concept* ψ is a function which, given a TU game in $\mathcal{G}^n$, selects a probability distribution over $\mathbb{R}^n$, i.e.

$$\begin{aligned} \psi : \mathcal{G}^n &\to \Delta\left(\mathbb{R}^n\right) \\ G &\mapsto \psi(G) \end{aligned}$$

and we denote by $\psi\left(G|a\right)$ the *probabilistic solution set* of the conditional n-players TU game $G|a$. Given a conditional TU game $G|a \in \mathcal{G}^n$, if an allocation $\pi^a \in \mathbb{R}^n$ belongs to the support of $\psi\left(G|a\right)$ then we say that π^a is a *solution* of the conditional game $G|a$ and we denote it by π^{a^*}.

At this point, we make the key conceptual move of remarking that any set of allocations that are solutions of a given TU game in $\mathcal{G}^n$, can be identified with the uniform distribution defined over the set itself. Building on this intuition, we observe that we can identify allocations with n-dimensional vectors of real-valued random variables[9]. Accordingly, we denote by Π_i^a the random allocation of value to player $i \in I$ in the conditional game $G|a \in \mathcal{G}^n$ and by $\Pi^a = \left(\Pi_1^a, \ldots, \Pi_n^a\right)$ the corresponding random allocation vector.

4.4 Evaluation of Generalized Biform Games

We now impose assumptions on players' preferences such that players evaluate outcomes of the cooperative stage game by a convex combination of three elements: the maximum and the minimum possible payoffs prescribed by the interval implied by the solution concept chosen (usually the core) and a third term that represents the value that, on average, can be allocated to the player in the conditional game. This latter term synthesizes one of the main novelties of our setup with respect to the seminal biform games framework: by embedding a probability distribution over the set of solution allocations, we allow the framework to capture more implications of the competitive structure on the possibilities of value capture of the players, while maintaining backward compatibility with the Brandenburger and Stuart (2007) framework.

[9] We recall that, given a probability space (Ω, Σ, P) and the measurable space $(\mathbb{R}, \mathcal{B})$, a *real-valued random variable* X is a measurable function $X : \Omega \to \mathbb{R}$.

Formally, let $\chi_i(I;a)$ be the appropriability interval of player $i \in I$ in the conditional game $G|a$. Then

$$u_i(\chi_i(I;a)) := \gamma_i \min \ \chi_i(I;a) + (1-\gamma_i-\delta_i)\,\mathbb{E}\left[\chi_i(I;a)\right] + \delta_i \max \ \chi_i(I;a) \tag{4.1}$$

where γ_i and δ_i are two non-negative real numbers such that $\gamma_i + \delta_i \in [0,1)$, $\mathbb{E}\left[\chi_i(I;a)\right]$ denotes[10] the expected value of the allocation to player $i \in I$ within the interval of value capture $\chi_i(I;a)$, where expectation is taken with respect to the objective chances of appropriation of each conceivable value to player i and $\min \chi_i(I;a)$ and $\max \chi_i(I;a)$ denote the minimum and the maximum of the interval of value capture $\chi_i\,(I;a)$. Essentially, $\mathbb{E}\left[\chi_i(I;a)\right]$ represents the expected value capture over the set of possible allocations across all players, and $\gamma_i + \delta_i \in [0,1]$ the indices representing respectively the subjective weight given to the most pessimistic outcome (γ_i) and the subjective weight given to the most optimistic outcome (δ_i) while $(1-\gamma_i-\delta_i)$ is the weight given to the more objective probabilistic outcome. Together, they allow to flexibly represent different levels of overconfidence (high δ_i), under-confidence (high γ_i), while accounting for the full geometry of the set of possible allocations ($\mathbb{E}\left[\chi_i(I;a)\right]$). Whenever the player evaluates $\chi_i(I;a)$ according to (4.1) we say that it does so by considering its *generalized expected appropriation.*

This suggests another feature of our solution: the observation of deviations from "objective" (i.e. *frequentist*) expected appropriation reasoning can be immediately related with over- or under-confidence considerations. In other words, expected allocation reasoning provides the possibility of isolating an objective component of the players' assessment of the value of a coalition, which is computed by considering the relative frequency with which any give value is allocated to a player, from a subjective component, which embeds subjective distortions of appropriation chances.

It is immediate to observe how this representation includes as a limit case the one of Brandenburger and Stuart (2007). Specifically, in the present work, we assume that preferences

[10] We use this notation instead of the more precise $\mathbb{E}\left[\Pi_i^a|\Pi^a\right]$ with $\Pi^a\left(\omega\right) \in R$, for all $\omega \in \Omega$, where R is a generic set of restrictions on players' allocations that embeds the assumptions for the problem under analysis. Although improper, we choose this notation for simplicity and ease of interpretation.

over cooperative-stage outcomes are represented by the Non-Extreme Outcomes Expected Utility (NEO-EU) criterion of Webb and Zank (2011) with linear utility.

We remark that, although we do not pursue this direction here, our definition of generalized biform games further allows for the embedding of more general preference structures on the set of outcomes of the cooperative stage game. In this context, both this criterion and the one used by Brandenburger and Stuart (2007) can be seen as special cases of preference representations.[11] In fact, the latter can be seen as the requirement that $\succsim^*$ be represented by the Hurwicz (1951) criterion[12] over the core outcomes, that is, by a convex combination of the extremes of the projections of the core, for every player.

This suggests another feature of our setup: in what follows we will posit that the evaluation of the cooperative stage payoffs entails some form of expected value reasoning.[13] This will imply the set of solution concepts be coherent with this assumption, which is compatible with a wide array of choices, as we will illustrate further along. However, this is a consequence of the axiomatic system imposed upon the preference relation $\succsim^*$ which, we note, can be made compatible with non-expected value reasoning-based solution concepts such as, for example, the nucleulous.

As implemented in the present paper, our framework thus generalizes Brandenburger and Stuart (2007) in two main directions: *(1) we consider all outcomes in a value capture interval together with their chances of appropriation and (2) we allow for cooperative-stage solution sets that do not necessarily coincide with the core.* In fact, at this level of generality, our framework is silent with respect to the solution concept to be adopted in the cooperative stage of the biform game. This allows the present setup be compatible also with the use of

[11] In mathematics, a representation theorem is a theorem that states that every abstract structure with certain properties is isomorphic to another (abstract or concrete) structure. One of the cores of decision theory concerns the representation of preference relations $\succsim$ and studies behavioral conditions for observable preference relations to be equivalent to decision makers choosing as if they are maximizing functionals such as, for example, the one in (4.1).

[12] Hurwicz's (1951) criterion evaluates future outcomes under uncertainty by giving a weight to the worst-case scenario and the complement to the best-case scenario.

[13] The Hurwicz criterion can, in fact, be seen as a degenerate form of expectation. Axiom-wise, the main difference between the Hurwicz Expected Utility and the NEO-EU criterion is that, while the first one requires a form of exchangeability over a specific type of events, the latter requires the preference relation to satisfy forms of consistency on the evaluation of the most and least preferred outcomes in lotteries when probability shifts are involved. For more details, we refer the interested reader to Gul and Pesendorfer (2015) and Webb and Zank (2011).

recent developments in the theory of value-based strategy, such as the refinements of intervals of value capture based on different notions of competitive intensity in Montez, Ruiz-Aliseda, and Ryall (2017).

5 Expected allocations

Technically, the simple but crucial step that we take here is to endow the solution set with a probability distribution. This grants us the possibility of considering not only extreme feasible values, but also the full range of possible solutions for each player, together with the shape of the solution set. Hence, this evaluation reflects more fully how competition restricts value capture in the game.

We depart from Brandenburger and Stuart (2007) and assume that each player evaluates his appropriability range by his *(generalized) expected appropriation*, instead of a convex combination of the extremes of the value capture range, which coincides, for every player, with his coordinate projection of the solution set. As already remarked, the solution concept they adopted is the core, but this is not a necessity in our setup.

When we focus on non-negative efficient allocations, the analysis of allocation of value of n-players conditional TU games immediately translates to that of subsets of regular simplices with side length equal to the total value created by the grand coalition of players in the conditional game. In this case, a natural candidate for endowing the solution set with a probability distribution is the use of a (continuous) uniform distribution over regular simplices, although other choices are possible. This assumption, that we will maintain throughout this paper, formalizes the idea that, ex-ante, all allocations in the simplex are equally possible. Given our hypotheses, the coordinates of the simplex are jointly distributed according to a Dirichlet with dispersion parameter equal to 1 for all $i \in I$. The specific choice of solution concept, in turn, induces different probabilities over the set of allocations of value created, by restricting its support.

The generalized expected appropriation comprises of two main terms: the first one is an expectation of the value capture, computed with respect to a frequentist probability distribution, the other term represents a relative (upward or downward) distortion of this value. This decomposition allows to neatly separate the objective assessment of chances of value capture, which are represented by the expected value term, from the subjective assessment of these chances, which are reflected by the evaluation weights that players place on the extreme values of their value capture range. The expectation of the value capture is computed, for every player, by weighting each outcome in the range by the relative proportion of allocations that are solutions of the conditional game which assign that outcome to the focal player. We label this expectation term as *objective* because the expectation term is computed by setting probabilities as relative frequencies of value capture. This way, the difference between the objective expected value capture and the generalized expected appropriation of each player uniquely determines the weights placed on the extremes, which represent the *subjective* distortion of appropriation chances for the player. This feature of our modeling solution allows to therefore interpret the weights assigned to the extremes values of the interval of value capture as overconfidence indices.

Leaving aside the above behavioral distinction, the consideration of an expected allocation to each player, computed by taking into account his chances of value capture within the coalition, allows to take into account richer considerations on the structure of the competition in the outcomes of the game. In particular, a variation in the number of value capturing players in a coalition will induce a variation in the chances of appropriation of value of the players in that coalition. This will imply that, coherently with the intuition, a player may not be indifferent between two structures of competition, even when they induce the same interval of value capture for that player.

Given that expected value based evaluation criteria are especially compelling when solutions are set valued, in what follows we will review some set-valued solution concepts and the strategic assumptions underlying them. Then we will move on to the more operational side of computing these expectations and observe properties and implications of the use of these quantities.

5.1 Set Valued Solutions for Stable Coalitions: from the Core to the Weber Set

The core of a game holds a special status within the theory of cooperative game due to its strong intuitive meaning: it is the set of allocations that no coalition can improve upon. In other words, it is the set of outcomes such that no player has incentives to leave the coalition. Yet the use of the core has been often criticized in strategic management. The main critiques to the use of the core boil down to two arguments: on the one hand, the core *may not exist*, frustrating analysis beyond this finding, whereas, on the other hand, when it exists, *it may be too big*, suggesting an uncomfortable indeterminacy. The latter point is tackled in this paper by considering a natural point-selection within the set, as we will investigate at depth in the next subsection.

The possible non-existence of the core has been tackled by strategy scholars by suggesting the use of alternative solution concepts such as the nucleolus and the Shapley value (Ross, 2018). However, it must be remarked that these solutions have profoundly different interpretations in terms of strategic behaviour with respect to the core. The lack of existence of the core is per se signaling the impossibility for stability of coalitions to occur under given circumstances. Following the value added principle, a natural solution concept is given by the *reasonable set*, that is the set of all pre-imputations that give no player more than the largest amount that he can contribute to the coalition (Milnor, 1952).

If one is to maintain the stability of coalitions as a northern light in the quest for a good candidate solution concept, then it is possible to resort to alternative set-valued solution concepts that are closer in spirit to the core. Let us preliminary consider the concept of the *excess of a coalition S at allocation π* as the gain to the coalition S if its members depart from an agreement that yields π in order to form their own coalition.

Let $\varepsilon \in \mathbb{R}$, the *strong ε-core* is the set of efficient payoff vectors that cannot be improved upon by any coalition if forming a coalition entails a cost of ε. While the strong ε-core is always non-empty (that is, there always exists an $\varepsilon \in \mathbb{R}$ such that the set is non-empty), the core corresponds to the case $\varepsilon = 0$.

A related concept is that of *least-core*, that is defined as the smallest non-empty strong

ε-core (Maschler et al. 1979). An interesting property of the least core is that, if the core of a game is not empty, then the least-core is a located within the core, whereas, if it it is empty, then the least-core may be regarded as revealing the latent position of the core, by indicating the minimum cost of deviations that must be imposed on players in order to maintain stability of coalitions.

The *stable set* (Morgernstern and VonNeumann, 1953) characterizes the nonempty set K of imputations such that for all $\pi^a, \rho^a \in K$, no π^a dominates ρ^a and no ρ^a dominates π^a (internal stability), and for all $\sigma^a \in \mathcal{I}(a) \setminus K$ there is an imputation $\pi^a \in K$ such that π^a dominates σ^a (external stability).

A concept related to both the stable set and the core is that of *subsolution* (Roth, 1976). A subsolution of a game is a nonempty set L of imputations such that its elements are internally stable; if $\pi^a \in L$ and ρ^a dominates π^a, then $\rho^a \in Dom(L)$; and if $\pi^a \notin L \cup Dom(L)$, then there is an imputation $\rho^a \notin L \cup Dom(L)$ such that ρ^a dominates π^a. It is well known that nonemptiness of the core implies nonemptiness of subsolution sets. In addition, the intersection of all subsolutions is a subsolution known as the *super core*. The relation among the solution concepts is due to the fact that it is always the case that the core satisfies the first two requirements characterizing subsolutions. The additional requirement for subsolutions guarantees a stronger stability to the core allocations in that all imputations outside the core which are also not dominated by the core are dominated by some imputation having the same property. This implies that, when the core coincides with the super core, the only stable subset of imputations lies in the super core.

Some forms of stability of the outcome set applies under some standard hypotheses (super additivity or convexity of the game) also to the set of marginal contribution vectors, that is the *Weber set* (Rafels and Tijs, 1997). The use of the Weber set as candidate solution is especially justifiable when considerations about average marginal contributions are paramount. Among its properties, it also is well known that, when the core is not-empty, the Weber set is a super-set of the core.

Solution Sets under Bounded Cognition

Solution sets, besides resulting from the application of conditions imposed on allocations of value to players, implicitly incorporate requirements about the players' knowledge of the competitive environment. For example, in strategic behavior settings, positing that solution allocations belong to the core is implicitly equivalent to requesting all players be able to know or anticipate the value created by all possible sub-coalitions in the competitive arena. While this requirement, adhering to the classic Economic tradition, represents a useful benchmark, it may ascribe an unrealistic degree of cognition to players. In particular, it may be the case that some players are not aware of the value created by some of the possible sub-coalitions of players in the game. Solution sets that incorporate these requirements for allocations of value to players will result in super-sets of the core. The analysis hereby developed extends smoothly also to these cases. Although we do not explore this direction in detail here, we remark that incorporating epistemic considerations in biform games, both at the cooperative and non-cooperative stage(s), would represent a meaningful advancement of the present work (Aumann and Brandenburger, 1995: Battigalli and Siniscalchi, 1999; Menon, 2018).

5.2 Centroids: from the Core-Center to the Shapley value

In this subsection, we will focus on the interpretation and computation of $\mathbb{E}\left[\chi_i(I;a)\right]$, the expected value of value capture for player $i \in I$.

Given the observation that we can see the coordinates of the $(n-1)$-dimensional simplex as n random variables, the question of computing $\mathbb{E}\left[\chi_i(I;a)\right]$ is the same as that of computing the expected value of the i-th coordinate of the solution set in $\mathbb{R}^n$, which coincides with the i-th coordinate of its centroid in $\mathbb{R}^n$.[14] Operationally, as we will observe next, computing expected allocations of value to players is particularly immediate when the solution set reduces to special cases of regular polytopes. In the general case, in order to take into account

[14]Recall that the *expected* or *mean value* of a continuous random vector X with joint PDF f_X is the *centroid* of the probability density, i.e.

$$\mathbb{E}\left[X\right] = \int_{-\infty}^{\infty} x f_X(x) dx$$

In other words, the centroid is the center of gravity of the distribution of X.

arbitrary bounds to value capture, we must consider the subset of Δ^{n-1} generated by the restrictions corresponding to the solution concept selected.[15] A general feature of centroids is the fact that they inherit all properties of the set to which they belong, in addition to properties of balancedness (or fairness) that descend directly from the definition of centroids. This makes the centroid of a set-valued solution a good candidate for a point-solution with given desirable properties.

If we consider the set of all coalitionally rational and efficient allocations, i.e. the core, then the expected allocation of value to player i corresponds to the i-th coordinate of the centroid of the core, or *core-center*, whose properties have been studied in González-Díaz and Sánchez-Rodríguez (2003a).

Expected allocations: the simplicial case

Let us start our analysis of expected allocations by investigating the case of solution sets of conditional n-player games that can be represented by simplices in $\mathbb{R}^n$, while we will address the case of general solution sets next. When the solution set is a simplex, computing its centroid, whose coordinates correspond to the expected allocation for each player, is immediate. In fact, in this case, the centroid is found by simply averaging the vertices' coordinates of the simplex. Formally, let $v_1, \ldots, v_m$ denote the vertices of an $(m-1)$-simplex, where each v_j, with $j = 1, \ldots, m$ is an m-dimensional vector in $\mathbb{R}^m$. Then, the centroid C is

$$C = \frac{1}{m} \sum_{j=1}^{m} v_j$$

The next example illustrates the case when each player in the coalition of all players can appropriate value in the whole range between their reservation value and the total value created by the coalition. To illustrate our point we will exemplify strategic situation that make use of the core as solution concept. However, we underline that our considerations apply to all those cases in which the solution concept shares the relevant geometric properties of our results. In particular, here our results apply anytime the solution set is a simplex.

[15] Note that, for any $C \subset I$, the subset of the simplex generated by the restrictions in R_C is a convex polytope, yet not necessarily a simplex.

Example 3. Let us start by considering the core of a conditional TU 3-players game such that, for all $i, j \in I$, $W_a(\{i\}) = W_a(\{i,j\}) = 0$ and, for all $i \in I$, $av_i(I) = W_a(I) = 1$. This set coincides with the standard 2-dimensional simplex Δ^2. In this case, we obtain that, for every player $i \in I$, the interval of value appropriability χ_i^0 is the interval $[0, 1]$, that is, all allocations within the simplex Δ^2 are possible. As per our standing assumption, we recall that within the solution set no allocation is objectively more likely than the other. This translates into the fact that our (probabilistic) solution set here coincides with a uniform distribution over Δ^2. Here, since, by hypothesis, mass is uniformly distributed over the whole simplex, the expected allocation of value to the players coincides with the centroid of the triangle in Figure 5.1.

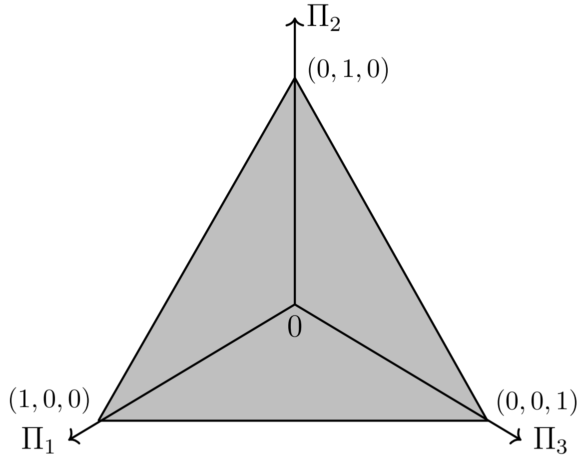

Figure 5.1: The 2-dimensional standard simplex Δ^2

It is immediate to compute that the core-center is $\mathbb{E}(\chi_i(I; a)) = \frac{1}{3}$ for all $i \in I$. Therefore, in this simple case, where all players can create no value outside the grand coalition yet, when coalescing, have the potential of capturing all value created, we have that the expected value of the appropriability interval for each player is equivalent to a fair distribution of the value created. □

Expected allocations: the general case

Now we move on to the general case where the solution set can be represented by a convex subset of a regular simplex in $\mathbb{R}^n$. This situation is by far the most common in application, as, even when considering the core as a solution set, this is often neither a singleton, nor a simplex, if not in very special cases. In this case, the expected value of the allocation vector π^a is still a centroid, but of the constrained region defined by the restrictions implied by the solution concept on the set of allocations of the conditional game $G|a$. In particular, for the core, we remind these restrictions be feasibility, coalitional rationality and efficiency.

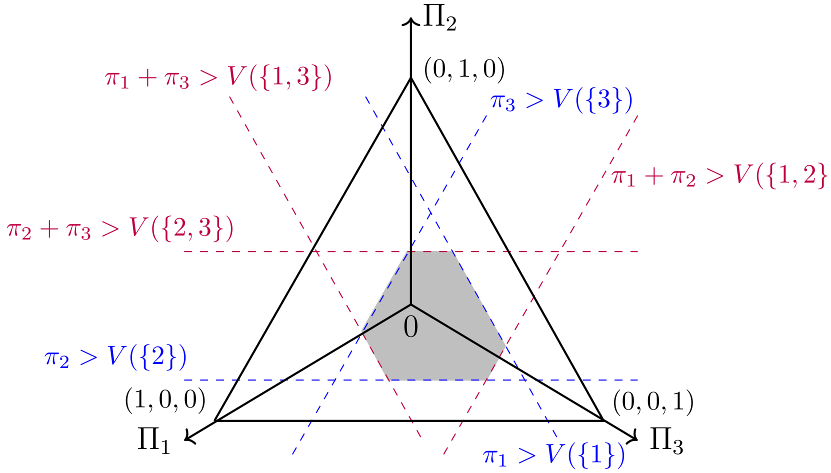

Figure 5.2: A restricted region R (grayed out) in Δ^2

Once the constrained region R is identified, we can obtain its centroid by computing the conditional expected value of the vector coordinates over the constrained region. Formally,

$$\mathbb{E}\left[\Pi_i|\Pi \in R\right] = \frac{\int_R \pi_i f_\Pi(\pi) d\pi}{\int_R f_\Pi(\pi) d\pi} \mathbb{I}_{\left\{\sum_{i=1}^n \pi_i = 1\right\}}(\pi)$$

Geometrically, R is a convex polytope in $\mathbb{R}^n$ given by the finite intersection of the half-spaces generated by the restrictions defined by the solution concept selected, on the pre-imputation set of the conditional game $G|a$. The vertices of this convex polytope can be computed by using the McLean and Anderson (1966) algorithm for calculating the coordinates of the extreme vertices of a constrained region. Hence, the expected allocation vector for the conditional game $G|a$ can alternatively be computed via geometric methods.[16]

[16]For procedural details, see, for example, Kaiser and Morin (1993).

Example 4. Let us consider a variation of our introductory example of "ecosystem paradox". Assume four firms: an operating system firm (OS), an application software firm (A), and two microprocessor firms (C_1 and C_2). Value can only be created with the OS, the *App* and one of the microprocessor firms. The two chipmakers firms are substitute to each other, but an additional k of value can be created with C_2's chip on top of the baseline value of 1 that can be created with C_1. The characteristic function is thus:

S	$v(S)$
$\{OS, App, C_1, C_2\}$	$1 + k$
$\{OS, App, C_1\}$	1
$\{OS, App, C_2\}$	$1 + k$
Any other subset of players	0

Any allocation of value can be written in barycentric coordinates, $(\pi_{OS}, \pi_{App}, \pi_{C_1}, \pi_{C_2})$. In this game, the core has a trapezoid shape with the following extreme points, listed clockwise:

Extreme point	π_{OS}	π_{App}	π_{C_1}	π_{C_2}
A	1	0	0	k
B	0	1	0	k
C	0	$1 + k$	0	0
D	$1 + k$	0	0	0

Calculating the barycentric coordinates of the centroid of the core (see Appendix B), we find it corresponds to the following allocation of value:

$$\pi_{OS} = \pi_{App} = \frac{k^2 + 3k + 3}{3k + 6}, \pi_{C_1} = 0, \pi_{C_2} = \frac{k(3 + k)}{3(2 + k)}.$$

□

This case allows to embed into our analysis the idea that different strategic requirements on allocations imply different restrictions on the set of feasible solutions of a conditional game. In particular, being usually defined as intersections of half-spaces, solution sets in cooperative games tend to maintain the property of being convex polytopes. This permits our technique be compatible with different solution concepts in cooperative games.

In particular, as previously discussed, the Weber set represents a reasonable solution set when focus is put on the totality of the marginal contributions of the players in the game, especially in situations where order of entrance in the market is ex-ante unknown. An interesting and well-known property of Weber sets is that its centroid is the Shapley value (González-Díaz and Sánchez-Rodríguez, 2003b), a solution concept often invoked in value-based strategy when the core is empty in that this quantity always exists (Ross, 2018). Our centroid-based approach to the analysis of biform games thus reconciles the use of these two different solutions concepts. This result further suggests an additional way to easily compute the centroid of a core, for all those games in which the core and the Weber set coincide, that is, for the class of convex games (Peters, 2015 : Theorem 18.6). More importantly, this perspective allows to add a further strategic justification for the use of the Shapley value, whose characterization is strongly rooted in fairness considerations as opposed to competition dynamics that are instead dominant in the case of the core, in wide classes of games.

An application: decreasing returns to value added

Our framework further allows to answer questions related to the elasticity of value capture to the competitive environment. In other words, in applied models of value-based business strategy, one central question is that of understanding how value appropriation changes as value creation changes (García-Castro and Aguilera, 2015; Lieberman et al. 2017; 2018). It is immediate to observe that this type of questions can be effectively answered by studying the behavior of the expected value capture of players. Along this line, the next result is, perhaps, the most surprising within the present work. In particular, Proposition 1 shows that there are decreasing marginal relative returns from value creation and that this property is invariant to the number of incumbent value capturing players. That is, the relative share of value added that a player can expect to appropriate is maximal when the value added is smallest. In applications, this property is of fundamental importance in understanding the incentive structure of bottleneck players in ecosystems (Chatain and Plaksenkova, 2020).

Proposition 1. *Let $I = \{1, ..., n\}$, with $n \in \mathbb{N}$ such that $2 < n < \infty$, denote the set of players with $i \in I$ denoting the i-th player and $-i$ denote all players $j \in I \setminus \{i\}$, that is all* *players except the i-th player. Let Π_i further denote the value captured by player i and be such that $a_i \leq \Pi_i \leq b_i$ for all $i \in I$. The set of restriction on players' values appropriation is $R = \times_{i \in I} [a_i, b_i]$ and $\Pi = (\Pi_i)_{i \in I}$ is the (random) vector of value capture for all players in I, with generic realization $\pi = (\pi_i)_{i \in I}$. Let us assume that $a_{-i} = a_i = 0$, $b_{-i} = 1$ and $b_i \in (0, 1)$. Then, for player i, it holds that*

$$\frac{\mathbb{E}(\Pi_i | \Pi \in R)}{b_i} \longrightarrow \frac{1}{n} \quad as \quad b_i \longrightarrow 1$$

and

$$\frac{\mathbb{E}(\Pi_i | \Pi \in R)}{b_i} \longrightarrow \frac{1}{2} \quad as \quad b_i \longrightarrow 0$$

Example 5. Let us consider again Example 4 and focus on returns to added value. We remind that the expected allocation of value to players is as follows:

$$\pi_{OS} = \pi_{App} = \frac{k^2 + 3k + 3}{3k + 6}, \pi_{C_1} = 0, \pi_{C_2} = \frac{k(3 + k)}{3(2 + k)}.$$

Clearly, the weaker chipmaker (C_1) cannot capture value in any way since its added value is always zero. If $k = 0$, it perfectly substitutes for C_2, leading both to capture no value. C_2 can at most capture up to its added value, k. When k increases, the total pie increases accordingly. But how is split, at the margin, this extra value? We see that the added value of C_2 is equal to k. According to Proposition 1, as the added value of C_2 increase above zero, C_2 should capture exactly $\frac{1}{2}$ of the increase. In this example, the quantity of interest is the marginal increase in π_{C_2} at $k = 0$. Accordingly, we find:

$$\frac{d\pi_{C_2}}{dk} = \frac{k^2 + 4k + 6}{3(2 + k)^2}, \text{ and } \lim_{k \to 0} \frac{d\pi_{C_2}}{dk} = \frac{1}{2}.$$

□

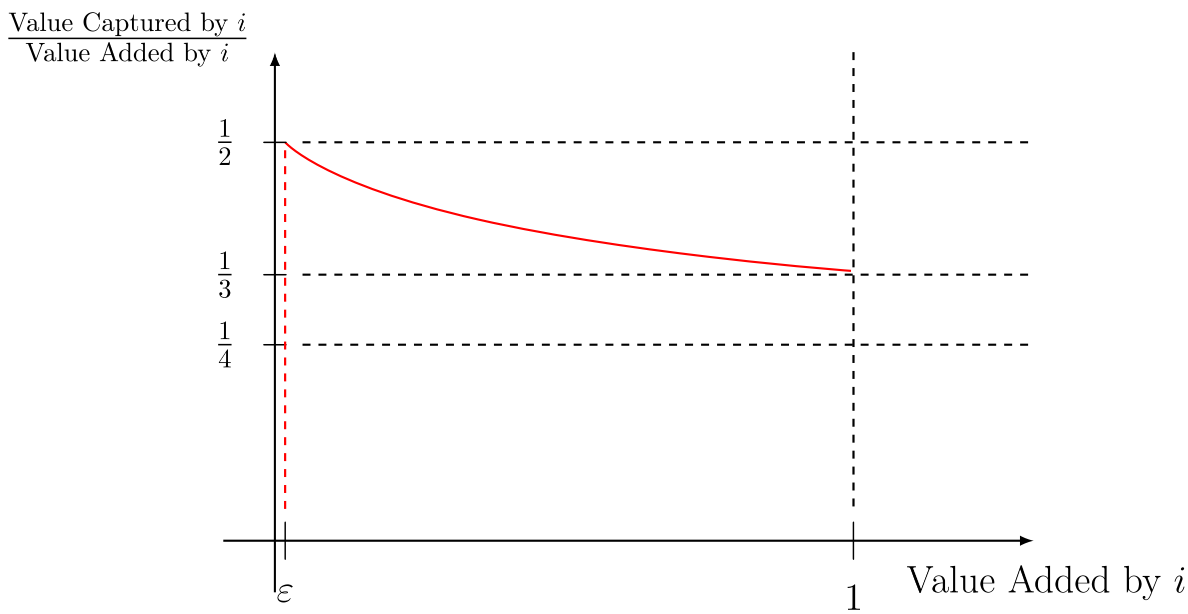

Figure 5.3: Player i's returns to value capture as a function of value added

We further remark that the technique used to prove Proposition 1 provides an alternative way of computing the centroid of a solution set, when the latter can be described as a difference between two regular simplices.

6 Concluding remarks

We provided an extension of the seminal biform games setup of Brandenburger and Stuart (2007) that, while maintaining all of its desirable features, such as the absence of constraints upon the analysis of competition in the coalitional stage, further allows to overcome some of the limitations of the original biform setup.

We remark that one main innovation introduced in this paper regards the possibility of adopting a more general representation of preference relations over cooperative-stage payoffs. Hence, our paper complements the work in value based business strategy that focuses on verifying the impact of imposing richer assumptions on the relation between players (Bryan et al, 2019; Chatain and Zemsky, 2011; Ryall and Sorenson, 2007).

By adding considerations based on expected value reasoning, we answer to the often raised criticism related to the possible indeterminacy of the solution when the core is used as solution concept. The use of the centroid of the set-valued solution allows to select a point-solution that maintains all desirable properties of the chosen set-valued solution concept, together with additional properties of balancedness. Differently from the use of the Shapley value as point-solution concept, our proposed evaluation criterion can be justified by competitive dynamics instead of fairness concerns. Furthermore, as we have observed both though theory and examples, our solution further addresses issues related with invariance to the competitive structure of the game, which can arise when the seminal biform game setup is used.

The criterion we specified to represent preferences over cooperative-stage outcomes can be seen as a genuine generalization of the one of Brandenburger and Stuart (2007), which arises as a special case of ours when the core is considered as solution concept and players in their evaluation only focus on extreme outcomes instead of adding expected value considerations. This latter criterion, like the one adopted in the seminal biform games' approach, is still axiomatically justified by behavioral assumptions on the players' preferences as it coincides with a case of the Non-Extreme Outcomes Expected Utility criterion of Webb and Zank (2011). In addition, the possibility of isolating an objective expectation component from the evaluation criterion allows a more transparent analysis of confidence considerations, which represent can be identified as a subjective distortion of appropriation chances on behalf of the players. However, we remark that both the Hurwicz (1951) criterion, used by Brandenburger and Stuart (2007), and the Webb and Zank (2011) criterion, on which we focused here, are both special cases of preferences' representations and, as such, are both instances of generalized biform games. Hence, although in the present work we suggest the use of a specific evaluation criterion for cooperative-stage payoffs that is immediately backward compatible with the original biform games' setup, the framework that we outlined allows for the use of more general evaluation criteria, through the embedding of a general preference relation within the biform structure. Different behavioral requirements can be imposed upon the preference relation, leading the way to the analysis of additional behavioral considerations, such as attitudes towards uncertainty, as advocated for in Gans and Ryall (2017).

Finally, we remark that our framework does not necessarily prescribe the use of the core as a solution concept. This feature allows, on the one hand, to make use of the recent developments in terms of intervals of value capture that have been studied in the value-based strategy literature. Relatedly, our approach allows to exogenously impose intervals of value capture and derive the restrictions on expected allocations that they imply. This approach can be seen as the converse of the more classical procedure of selecting a solution concept and obtain the implied intervals of value capture that are compatible with the requirements on allocations that characterize the chosen solution concept. Both approaches are compatible with our setup, conditional on maintaining the assumption that allocations belong to the imputation set of the conditional games. On the other hand, it leads the way to a structured analysis of the effects of different limits to strategic cognition, which has been recently advocated for in the competitive strategy literature (Menon, 2018).

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Appendix A: Proof of Proposition 1

Let N be a natural number greater than 2. The $(N-1)$-dimensional unit simplex $\Delta^{(N-1)}$, corresponding to the case where $a_j = 0$ and $b_j = 1$ for all $j \in I$ has area $V^1 = \frac{\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}$ and centroid with coordinates $C^1 = \left(\frac{1}{N}\right)_{i=1}^{N}$ in $\mathbb{R}^N$. Instead the polytope corresponding to the complement in $\Delta^{(N-1)}$ of the restriction of $\Delta^{(N-1)}$ generated by adding the constraint $1 \geq b_i > 0$ is a regular $(N-1)$-simplex with side length $(1-b_i)$ and thus area

$$V^2 = \frac{(1-b_i)^{(N-1)}\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}$$

and centroid coordinates

$$C_i^2 = \frac{1}{N}\left(1 + (N-1)b_i\right)$$
$$C_{-i}^2 = \frac{1}{N}\left(1 - b_i\right)$$

Therefore, the centroid of the restricted region is given by

$$C_i^* = \frac{V^1 C_i^1 - V^2 C_i^2}{V^1 - V^2} = \frac{\frac{\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}\frac{1}{N} - \frac{(1-b_i)^{(N-1)}\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}\frac{1}{N}\left(1+(N-1)b_i\right)}{\frac{\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}} - \frac{(1-b_i)^{(N-1)}\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}}$$

and

$$C_{-i}^* = \frac{V^1 C_{-i}^1 - V^2 C_{-i}^2}{V^1 - V^2} = \frac{\frac{\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}\frac{1}{N} - \frac{(1-b_i)^{(N-1)}\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}\frac{1}{N}\left(1-b_i\right)}{\frac{\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}} - \frac{(1-b_i)^{(N-1)}\sqrt{N}}{(N-1)!\sqrt{2^{(N-1)}}}}$$

Finally, since $C_i^* = \mathbb{E}\left(\Pi_i | \Pi \in R\right)$, we compute

$$\lim_{b_i \to 1} \frac{\mathbb{E}\left(\Pi_i | \Pi \in R\right)}{b_i} = \frac{1}{N}$$

and

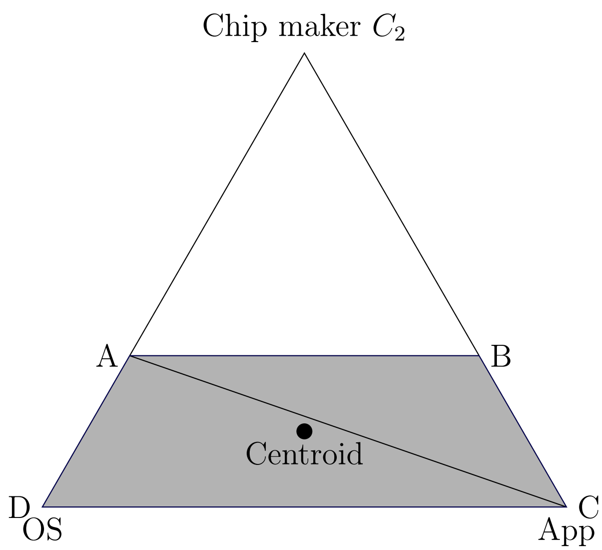
$$\lim_{b_i \to 0^+} \frac{\mathbb{E}\left(\Pi_i | \Pi \in R\right)}{b_i} = \frac{1}{2}$$

■

Appendix B: Core-center computation for Example 4

Since C_1's value capture is always zero, we work in the 3-dimension simplex defined solely by the allocations to players OS, App, and C_2.

To find the centroid of the trapezoid [ABCD], we first split it in two triangles, [ACB], and [ADC]. For each triangle, we find its centroid and calculate its signed area (the area has a positive sign when the summits are listed in an anticlockwise fashion). The centroid of the trapezoid is the barycenter of the triangles's centroids, weighted by their area.



The first triangle, [ACB], has for centroid $(\frac{1}{3}, \frac{2+k}{3}, \frac{2k}{3})$, calculated as the average of the barycentric coordinates of its three summits. The second triangle, [ADC], has for centroid $(\frac{2+k}{3}, \frac{1+k}{3}, \frac{k}{3})$.

Areas of triangles in barycentric coordinates are calculated as follows. If the summits have coordinates $P_i(x_i, y_i, z_i)$, $i = 1, 2, 3$, then the signed area of triangle $[P_1, P_2, P_3]$, as a fraction of the simplex's area, is calculated thanks to the formula:

$$[P_1 P_2 P_3] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_1 & z_3 \end{vmatrix}.$$

Applying this to [ACB] and to [ADC], their respective areas are:

$$[ACB] = \begin{vmatrix} 1 & 0 & k \\ 0 & 1+k & 0 \\ 0 & 1 & k \end{vmatrix} = k + k^2, \quad [ADC] = \begin{vmatrix} 1 & 0 & k \\ 1+k & 0 & 0 \\ 0 & 1+k & 0 \end{vmatrix} = k + 2k^2 + k^3.$$

Giving the weight $\frac{[ACB]}{[ACB]+[ADC]}$ to the centroid of $[ACB]$, and the weight $\frac{[ADC]}{[ACB]+[ADC]}$ to that of $[ADC]$, we find the centroid of the trapezoid as:

$$\left(\frac{k^2+3k+3}{3k+6}, \frac{k^2+3k+3}{3k+6}, \frac{k(3+k)}{3(2+k)} \right),$$

which gives the expected value capture of respectively OS, App, and C_2.